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Quantization of symplectic orbits of compact Lie groups by means of the functional integral

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Dedicated to I.M. Gelfand on his 75th birthday

Abstract. *The functional integral for the quantization of the coadjoint orbits of the unitary and orthogonal groups is given by means of an explicit construction of the corresponding «Darboux» variables.*

INTRODUCTION

In this paper we describe a method for the quantization of the Hamiltonian action of compact group on its coadjoint orbit. In the spirit of the method of orbits [1, 2] this quantization is supposed to give the corresponding representation of the group. Our method is based on the functional integral form of the matrix elements of the representation.

The simplest case of the group $SO(3)$ was considered before [3]. We shall present it in the introduction to illustrate the underlying ideas of our general approach. In the main text we shall consider unitary and orthogonal groups. Symplectic and exceptional cases can be treated in analogous way and we plan to present the explicit formulas in a separate publication.

The main technical problem in the generalization of the $SO(3)$ example to the higher rank case consists in finding the «Darboux variables» for the canonical symplectic

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form on the orbit. We describe the derivation of such variables for unitary and orthogonal cases in §1 and §2, correspondingly. In §3 we use them to evaluate the functional integral by reducing it to finite-dimensional one.

The condition of quantization of the parameters of the orbit [1, 2] is used in a natural way in our formalism. Indeed, it allows to satisfy the requirement of single-valuedness of the expression $\exp\{i \times \text{action}\}$.

We prove that the representation obtained by our method coincides with that associated with the orbit by an explicit evaluation of its character. Of course the matrix elements and the character are calculated for the action of a chosen Cartan subgroup. This subgroup is associated in a natural way with the choice of the Darboux variables.

Now let us consider the illustrative case of the group $SO(3)$. Its orbits are realized as spheres S^2 parametrized by their radius. It is convenient to use for the beginning coordinates $x^1, x^2, x^3 : \sum_i (x^i)^2 = m^2$. The symplectic form Ω is given by

$$\Omega = \frac{1}{2m^2}\epsilon^{abc}x^a \mathrm{d}\, x^b \mathrm{d}\, x^c. \tag{1}$$

Functions x^a correspond to the generators of $SO(3)$ with the Poisson brackets

$$\{x^a, x^b\} = \epsilon^{abc}x^c.$$

Quantization condition for orbits leads to the requirement that m is an integer or a half-integer. The Darboux variables are given by the spherical coordinates

$$x^1 = m \sin\,\theta \cos\,\varphi,\ x^2 = m \sin\,\theta \cos\,\varphi,\ x^1 = m \sin\, \cos\,\theta,$$

so that

$$\Omega = m \mathrm{d}\,\varphi\, \mathrm{d}\, \sin\,\theta.$$

The 1-form $\omega = -\mathrm{d}^{-1}\Omega$, necessary to define the action, is not single-valued, because Ω is not an exact 2-form. If we choose

$$\omega = (\gamma + m\,\cos\,\theta)\,\mathrm{d}\,\varphi$$

then the action of the path

$$\varphi = \varphi(t),\ \theta = \theta(t), \qquad 0 \le t \le T,$$

is given by

$$S_0 = m\int_0^T \cos\,\theta \mathrm{d}\,\varphi + \gamma\int_0^T \mathrm{d}\,\varphi. \tag{2}$$

For infinitesimal closed contours around the singular points $\theta = 0, \pi$ this action is given by $2\pi(\gamma \pm m)$ so does do not contribute to the $\exp\{i \times$ action for integer values of $m \pm \gamma$. Thus we can choose $\gamma = 1/2$ if m is a halfinteger and $\gamma = 0$ if m is an integer.

The expression for the matrix element of the operator $\exp\{-ix_3 T\}$ will be taken in the form

$$(3) \qquad G(\varphi', \varphi'' = \langle \varphi''| e^{-ix^3 T} |\varphi'\rangle = \int \prod_t \mathrm{d}\,\eta(t) \mathrm{d}\,\varphi(t)\; \exp\{iS\},$$

where $\eta = \cos\,\theta$,

$$(4) \qquad S = \int_0^T (m\eta\dot\varphi - h)\mathrm{d}\,t + \gamma \int_0^T \dot\varphi \mathrm{d}\,t = S_0 - \int_0^T h \mathrm{d}\,t$$

and $h = x^3 = m\,\cos\,\theta$. Here the trajectories $\eta(t), \varphi(t)$ (over which we sum the function $\exp\{i \times \text{action}\})$ are subject to the condition

$$\varphi(0) = \varphi', \quad \varphi(T) = \varphi'' + 2\pi n$$

for any integer n.

Alternativly, this matrix element can be rewritten as a sum

$$(5) \qquad G(\varphi', \varphi'') = \sum_n G_0(\varphi', \varphi'' + 2\pi n),$$

where the matrix element $G_{01}(\varphi', \varphi'')$ is given by the same functional integral but with fixed boundary values $\varphi(0) = \varphi', \varphi(T) = \varphi''$. The variables $\eta(t)$ have values in the interval $-1 \le \eta \le 1$, whereas $\varphi(t)$ runs over the whole real axis $-\infty < \varphi(t) < \infty$.

The following comments to this definition are appropriate:

1. Expression (3) essentially coincides with the general formula for the functional integral on the phase space with the chosen «coordinate-like» variables [see formula 4]. The role of coordinates is played by φ in our case. The difference with the usual case of linear phase space in that our phase space is compact.

2. The phase space S^2 is replaced by the strip $-1 \le \eta \le 1, -\infty < \varphi < \infty$ which is a covering of the cylinder $-1 \le \eta \le 1, 0 \le \varphi < 2\pi$ i.e. of the sphere with two deleted points, which correspond to the singularities of our coordinate system. It is a common wisdom in quantum mechanics that one has to use the covering space in the functional integral.

3. The summation in (5) is an averaging over the corresponding discrete fundamental group, restoring the periodicity of the matrix element $< \varphi''| \exp\{-ix^3 T\}|\varphi' >$.

After these comments we shall calculate the path integral, using its definition by means of the finite dimensional approximation. We have

$$G_0(\varphi', \varphi'') =$$

$$= \lim (\frac{1}{2\pi})^N \int_k \prod \mathrm{d}\,\eta_k \mathrm{d}\,\varphi_k e^{i \sum_{k=1}^N \{m\eta_k(\varphi_k - \varphi_{k-1}) - m\eta_k \frac{T}{N} + \gamma(\varphi_k - \varphi_{k-1})\}},$$

(6) $$\varphi_0 = \varphi',\ \varphi_N = \varphi'',$$

where the number of integrations over η exceeds by one that φ. Integrals over φ give δ-functions which can be integrated over $\eta - s$. As a result we obtain

(7) $$\begin{aligned} G_0(\varphi', \varphi'') &= \\ &= \frac{1}{2\pi} \int_{-1}^{1} \mathrm{d}\,\eta e^{im\eta(\varphi'', \varphi' - T) + i\gamma(\varphi'', \varphi')} = \\ &= \frac{1}{2\pi} \left[\frac{e^{im(\varphi'' - \varphi' - T)}}{m(\varphi'' - \varphi' - T)} - \frac{e^{im(\varphi'' - \varphi' - T)}}{m(\varphi'' - \varphi' - T)} \right] \\ & e^{i\gamma(\varphi'' - \varphi')}. \end{aligned}$$

We can say that the whole functional integral is given by the contribution of the «semi-classical trajectories» $\eta(t) = \eta$.

To perform the averaging (5) we are to choose the regularization for the infinite sum

$$\sum_{-\infty}^{\infty} \frac{1}{2\pi n + \varphi'' - \varphi' - T}.$$

We shall use the following regularizations

(8) $$S^{\pm} = \lim_{\epsilon \to 0} \sum \frac{e^{\pm i\epsilon 2\pi n}}{2\pi n + \alpha} = \frac{e^{\pm \frac{1}{2} i\alpha}}{\sin \frac{\alpha}{2}}$$

and associate S^+ with the first term in (7) and S^- with the second. As a result we shall obtain

(9) $$\begin{aligned} G(\varphi', \varphi'') &= \\ &= \frac{1}{2\pi} e^{i\gamma(\varphi'', \varphi')} \frac{\sin(m + \frac{1}{2})(\varphi'', \varphi' - T)}{\sin \frac{1}{2}(\varphi'', \varphi' - T)} = \\ &= \frac{1}{2\pi} e^{i\gamma(\varphi'', \varphi')} \sum_{m_z = -m}^{m} e^{im_z(\varphi'', \varphi' - T)}. \end{aligned}$$

1. DARBOUX VARIABLES FOR THE UNITARY GROUP

In this section we shall reduce the symplectic 2-form on the coadjoint orbit $SU(n)$ to the manifestly canonical form, convenient for the functional integral.

In what follows we shall use a natural identification of coadjoint orbits with the adjoint ones. The point of the orbit has the form

$$M = gM_0\, g^+, \tag{12}$$

where $g \in SU(n)$ and M_0 is a fixed antihermitian traceless matrix. The symplectic form Ω can be written in the form

$$\Omega = \frac{1}{2} tr MYY, \tag{13}$$

where the 1-form Y with the values in the Lie algebra satisfies the equation

$$\mathrm{d}M = [Y, M] \tag{14}$$

and tr is the Killing-Cartan form. It is clear from (12) that $Y = \mathrm{d}g \cdot g^{-1}$ solves this equation; different choice of solution leads to the same expression for Ω.

We shall find explicit coordinates in which Ω has the Darboux form. Let $e^1, \ldots, e^n$ be an orthogonal set of eigenvectors of matrix M_0 with eigenvalues $2im_1, \ldots, 2im_n$ $m_1 \geq \ldots \geq m_n$ and

$$a^1 = ge^i. \tag{15}$$

Formulae (13) and (15) imply that Ω can be written as

$$\Omega = -\mathrm{d}\alpha,$$

$$\alpha = \frac{i}{2} \sum_1^n m_k[(a^k, \mathrm{d}a^k) - (\mathrm{d}a^k, a^k)] \tag{16}$$

where we use the complex scalar product.

If the complex variables a_i^k were independent, the form Ω would be given in explicitly canonical form in the linear phase of real dimension $2n^2$ which we shall call auxiliary space in the future. However, the variables a_i^k are subject to the orthonormality constraint of

$$\varphi_{ij} \equiv (a^i, a^j) - \delta^{ij} = 0 \tag{17}$$

This means that (16) is still not of the desired form and one must make a reduction with respect to these constraints [4].

The constraints (17) are naturally divided into two sets of the first and second class. The first set consists of the constraints

$$\varphi_i \equiv \varphi_{ii} = (a^i, a^i) - 1 = 0 \tag{18}$$

Indeed, their Poisson brackets on the auxiliary space vanish. Moreover, the constraints φ_i commute with the coordinate functions M_{ij} defining the orbit. This means that

$$\{\varphi_i, f(M)\} = 0$$

for any function on the orbit.

The constraints $\varphi_{ij}, i \neq j$ are of the second class in the generic case of distinct eigenvalues m_i. Indeed we have

$$\{\varphi_{ij}, \varphi_{kl}\}|_{\varphi=0} = i\delta^{il}\delta^{jk}\left(\frac{1}{m_i} - \frac{1}{m_j}\right) \tag{19}$$

and this matrix is nondegenerate in the subspace $i \neq j, k \neq l$. Thus the dimension of the physical phase space, i.e. the dimension of the auxiliary space minus twice the number of 1-st class constraints minus the number of 2-nd class constraints is given by

$$2n^2 - 2n - 2\frac{n^2 - n}{2} = n^2 - n.$$

This number coincides with the dimension of the maximal (nondegenerate) orbit given by the dimension of the group minus its rank. This means that the Hamiltonian reduction of the form (16) with respect constraints (17) leads to the Kirillov form on the orbit. The explicit reduction is realized by introduction of supplementary conditions in quantity equal to that of 1-st constraints, so that the matrix of all Poisson brackets of constraints and supplementary conditions is nondegenerate. The conditions

$$x^i \equiv a^i_n - \bar{a}^i_n = 0, \tag{20}$$

where a^i_n is the last component of a^i in a fixed basis, serve this purpose at a generic point. Indeed, we have the following Poisson brackets on the auxiliary space

$$\{x^i, x^j\} = 0, \{\varphi_{ij}, x^k\} = i\frac{\delta^{ik}}{m_i}a^j_n + i\frac{\delta^{jk}}{m_j}\bar{a}^j_n. \tag{21}$$

The last matrix in nondegenerate on the subspace $i \neq j$ (if $\operatorname{Re} a^j_n \neq 0$) which is sufficient for our statement.

Now, the form we are looking for is that induced by Ω on the submanifold $\varphi_{ij} = x^k = 0$. Our aim is to find explicit canonical (angle-action) variables for it. Making use of $x^i = 0$ we can rewrite α in the form

$$\alpha = \frac{1}{2}\sum m_k[(a_\perp^k, \mathrm{d}\, a_\perp^k) - (\mathrm{d}\, a_\perp^k, a_\perp^k)], \tag{22}$$

where $a_\perp^k$ are projections of a^k onto the subspace orthogonal to the last basic vector. In can be shown that this projection can be written as

$$a_\perp^k = \sum_{p=1}^{n-1} C^{kp} e'_p \tag{23}$$

with some moving orthogonal frame e'_p in this subspace and real coefficients C^{kp}, constituting a rectangular matrix C. The last condition leads to cancellation of terms with $\mathrm{d}\, C^{kp}$ in α which takes the form

$$\alpha = \frac{i}{2}\sum_{k=1}^{n}\sum_{p=1}^{n-1}\sum_{q=1}^{n-1} m^k C^{kp} C^{kq}[(e'_p, \mathrm{d}\, e'_q) - \mathrm{h.c.}]. \tag{24}$$

The matrix C satisfies the normalization condition

$$(a_\perp^i, a_\perp^j) = \sum C^{ip} C^{jp} = \delta^{ij} - a_n^i\, a_n^j; \tag{25}$$

along with the condition $\sum_{i=1}^{n}(a_n^i)^2 = 1$ it implies that the eigenvalues of the matrix M' with matrix elements

$$M'_{pq} = \sum C^{ip} m_i C^{jq} \tag{26}$$

vary in the domain (see [9])

$$m_i \geq m'_i \geq m_{i+1} \tag{27}$$

Due to its reality it can be diagonalized by a real orthogonal transformation which can be absorbed into the choice of matrix C This means that the form α now takes the form

$$\alpha = \frac{i}{2}\sum_{1=i}^{k-1} m'_i\, [(e'_i, \mathrm{d}\, e'_i) - \mathrm{h.c.}], \tag{28}$$

where e'_i are $h-1$ orthogonal vectors. Parametrizing e'_i as follows

$$e'_i = e^{-i\varphi'_i} a^{i'}, \quad Im a_n^{i'} = 0,$$

we get

$$\alpha = \sum_{1}^{n-1} m'_i \mathrm{d}\varphi'_i + \frac{i}{2} \sum_{1}^{n-1} m'_i [\,(a'_i, \mathrm{d}\, a'_i) - \mathrm{h.c.}]. \tag{29}$$

Now the second term in RHS of (29) looks exactly as (16) with the only change $n \rightarrow n-1$. We can repeat our procedure once more and so on. Finally, we get the expression we were looking for

$$\alpha = \sum_{1}^{n-1} m'_i \mathrm{d}\varphi'_i + \frac{i}{2} \sum_{1}^{n-2} m''_i + \ldots + m_1^{(n-1)} \mathrm{d}\varphi_1^{(n-1)}. \tag{30}$$

Here φ are angle variables, $0 \leq \varphi < 2\pi$, and the domain of $m_i^{(k)}$ is determined by our recurrent procedure of descent

$$m_i^{(k-1)} \geq m_i^{(k)} \geq m_{i+1}^{(k-1)}. \tag{31}$$

It is certainly worth mentioning that these coordinates constitute the classical analog of the Gelfand-Zetlin table with parameters of orbit $m_1, \ldots, m_n$ giving the first line. In a different context they were introduced in [10], however their connection with the symplectic structure on the orbits was not discussed.

In the course of deriving the final result (30) by means of the hamiltonian reduction we referred several times to the generic situation. In particular, matrix in (21) is singular when $\mathrm{Re}\, a_n^i = 0$ and reduction is to be continued in this case. This means that the coordinates $(m_i^{(k)}, \varphi_i^{(k)})$ which over the manifold $\prod \frac{n^2-n}{2} \times T^{\frac{n^2-n}{2}}$, where $\prod$ is polyhedron (31) and T-torus, are coordinates on the orbit with some «singular» set being deleted. The action which we introduce on the orbit in §3 is unsensible to this deletion mod 2π.

2. DARBOUX VARIABLES FOR THE ORTHOGONAL GROUP

The Lie algebra $SO(n)$ is generated by real antisymmetric matrices $M_{ij} = -M_{ji}$. The point of the orbit is given by

$$M = g M_0 g^T, \tag{32}$$

where $g \in SO(n)$ and M_0 can be taken in the form

$$M_{PF} = \begin{pmatrix} 0\, m_1 & & 0 \\ -m_1 0 & & 0 \\ & \ddots & \\ 0 & & 0\, m_r \\ 0 & & -m_r 0 \end{pmatrix}, \qquad n = 2r, \tag{33}$$

or

$$M_{PF} = \begin{pmatrix} 0\,m_1 & & 0 \\ -m_1 0 & & 0 \\ & \ddots & \\ 0 & & 0\,m_r \\ 0 & & -m_r 0 \\ & & 0 \end{pmatrix}, \quad \begin{matrix} n = 2r+1, \\ m_l \geq m_2 \ldots \geq m_r. \end{matrix}$$

We see that the orbit is parametrized by the numbers $m_1, \ldots, m_r$ (where $r = [n/2]$ is the rank of the algebra) which can be taken positive if $n = 2r + 1$, and all but one positive if $n = 2r$.

Let $f_1, \ldots, f_{2r}$ be a set of orthonormal vectors such that

$$M_{PF} f_{2k-1} = -m_k f_{2k},$$
$$M_{PF} f_{2k} = -m_k f_{2k-1}.$$

Then using the notation

$$g f_{2k-1} = g_k, \; g f_{2k} = p_k,$$

we have $\Omega = -\mathrm{d}\alpha$, where

$$\alpha = \frac{1}{2} \sum_1^r m_k [(p^k, \mathrm{d}q^k) - (q^k, \mathrm{d}p^k)] \tag{34}$$

and now our scalar product is real.

The real variables (p, q) span the auxiliary space with dimension $2rn$, and the form α must be obtained from (34) by hamiltonian reduction with respect to constraints

$$\begin{aligned} \varphi^{(1)}_{kl} &= (p^k, p^l) - \delta^{kl} = 0, \\ \varphi^{(2)}_{kl} &= (q^k, q^l) - \delta^{kl} = 0, \\ \varphi^{(3)}_{kl} &= (p^k, q^l) - \delta^{kl} = 0. \end{aligned} \tag{35}$$

The constraints

$$\varphi_i \equiv (p^i, p^i) + (q^i, q^i) - 2 \tag{36}$$

are of the first class, all others belong to the second class in a generic point $m_k \neq m_l, k \neq l$ with nonzero Poisson brackets

$$\begin{aligned} \{\varphi^{(3)}_{kl}, \varphi^{(1)}_{st}\}\,|_{\varphi=0} &= -\delta^{ls}\delta^{kt}\frac{1}{m_e} - \delta^{ls}\delta^{kt}\frac{1}{m_e}, s \geq t \\ \{\varphi^{(3)}_{kl}, \varphi^{(1)}_{st}\}\,|_{\varphi=0} &= -\delta^{ks}\delta^{lt}\frac{1}{m_k} - \delta^{kt}\delta^{ls}\frac{1}{m_k}. \end{aligned} \tag{37}$$

The number of physical degrees of freedom in the generic case of distinct m_i is given by

$$2rn - 2r - 2\frac{r^2 - r}{2} - r^2 - r = 2r(n - r - 1),$$

which coincides with the dimension of a nondegenerate orbit given by $\dim G - \operatorname{rank} G$.

The supplementary conditions needed for the reduction will be taken in the form

$$\chi^i \equiv p^i_n = 0. \tag{38}$$

Their Poisson brackets with the 1-class constraints

$$\{\chi^i, \varphi_k\} = \delta^{ik} q^i_n \tag{39}$$

are nonzero for a generic point $q^i_n \neq 0$.

To perform the reduction let us first consider the case $n = 2r + 1$. Let $q^i_\perp$ be the projection of q^i onto the subspace orthogonal to the last basic vector; $q^i_\perp$ form an r-dimensional subspace in the auxiliary space and can be expanded as

$$q^i_\perp = \sum c^{ik} e_k \tag{40}$$

with respect to some moving orthogonal frame e_k in this subspace; c^{ik} are real coefficients constituting an $r \times r$ matrix C. As in the unitary case differentials of this matrix C are absent in α due to respect to the condition $(p^l, e_k) = 0 = (p^l, q^k_\perp)$. Hence α takes the form

$$\alpha = \frac{1}{2}\sum m_i c^{ik}[(p^i, \mathrm{d}\, e_k) - (e_k, \mathrm{d}\, p_i)]. \tag{41}$$

The matrix $d_{ik} = m_i c^{ik}$ which appears in (41) can be written as

$$\mathrm{d} = u^T m' V, \tag{42}$$

where u and v are orthogonal matrices and m' is diagonal: $m' = \operatorname{diag}(m', \ldots, m'_r)$. If we denote

$$p^{i'} = up^i, \quad q^{i'} = vq^i, \tag{43}$$

then

$$\alpha = \frac{1}{2}\sum m'_i[(p^{i'}, \mathrm{d}\, q^{i'}) - (q^{i'}, \mathrm{d}\, p^{i'})]. \tag{44}$$

and here we can suppose that $m'_1 \geq m'_2 \geq \ldots m'_r$.

It is easy to prove that m'_i satisfy the condition

$$
\begin{aligned}
&m_i \geq m'_i \geq m_{i+1}, \\
&m_r \geq m'_r \geq -m_r.
\end{aligned} \tag{45}
$$

To prove this let us consider a symmetric positive matrix $\mathcal{D} = \mathrm{d}\mathrm{d}^T$ with eigenvalues m'^2_i. It follows from the definition of C that

$$\sum c^{ik} c^{jk} = \delta^{ij} - q^i_n q^j_n, \tag{46}$$

where q^i_n is the last component of q^i in the fixed basis. Thus

$$\mathcal{D} = M^2 - \Delta, \tag{47}$$

where M^2 is diagonal with eigenvalues m_i^2 and Δ is a positive matrix of rank 1 with matrix elements

$$\Delta_{ij} = m_i q^i_n m_j q^j_n. \tag{48}$$

This information is sufficient to state that [9]

$$m_i^2 \geq m'^2_i \geq m^2_{i+1}; \quad m_{r+1} = 0. \tag{49}$$

Generically we can choose such orthogonal matrices u and v that all eigenvalues m'_i but one will be positive. Thus we obtain the condition (45).

Set of vectors $p^{i'}, q^{i'}$ form an orthogonal moving basis in the $2r$-dimensional subspace of the n-dimensional vector space. We parametrize these vectors as follows

$$
\begin{aligned}
p^{i'} &= \hat{p}_i \cos\ \varphi'_i + \hat{q}_i \sin\ \varphi'_i, \\
q^{i'} &= \hat{q}_i \sin\ \varphi'_i + \hat{q}_i \cos\ \varphi'_i,
\end{aligned} \tag{50}
$$

where $\hat{p}^i, \hat{q}^i$ is some new orthonormal basis in a plane $p^{i'}, q^{i'}$ with the condition: $\hat{p}^i_{n-1} = 0$. Here $\hat{p}^i_{n-1} = 0$ is the last component of $\hat{p}^i$ in a fixed basis. In the new variables α takes the form

$$
\begin{aligned}
\alpha = &\sum_{i=1}^{r} m'_i \mathrm{d}\varphi'_i \\
&+ \frac{1}{2} \sum_{i=1}^{r} m'_i [\left(\hat{p}^{i'}, \mathrm{d}\hat{q}^i\right) - \left(\hat{q}^{i'}, \mathrm{d}\hat{p}'^i\right)].
\end{aligned} \tag{51}
$$

The second term in RHS of (51) looks as (34) with the change $n = 2r + 1 \mapsto n = 2r$, where n is dimension of vector space.

Now we briefly discuss the case $n = 2r$:

$$\alpha' = \sum_{1}^{r} m_i' (\hat{p}^i, \mathrm{d}\,\hat{q}^i) = \sum_{1}^{r} m_i' (\hat{p}_i, \mathrm{d}\,\hat{q}_\perp^i), \tag{52}$$

here $\hat{q}_\perp^i$ as above is the projection of $\hat{q}^i$ onto the subspace orthogonal to the last basic vector in the $2r$-dimensional vector space. But now $\hat{q}_\perp^i$ form an $(r-1)$-dimensional subspace in this space spanned by orthogonal moving vectors $\hat{e}_1, \ldots, \hat{e}^{(r-1)}$ and therefore

$$\hat{q}_\perp^i = \sum_{k=1}^{r-1} \hat{c}^{ik} \hat{e}_k. \tag{53}$$

The coefficient $\hat{C}^{ik}$ form an $r \times (r-1)$ rectangular matrix $\hat{C}$. In the $\hat{p}, \hat{e}$ variables we have

$$\alpha' = \sum m_i' \, \hat{c}^{ik} (\hat{p}^i, \mathrm{d}\,\hat{e}_k). \tag{54}$$

Now the matrix $\hat{\mathrm{d}} : \hat{\mathrm{d}}_{ik} = m_i' \, \hat{c}^{ik}$ can be written in the form

$$\hat{\mathrm{d}} = \hat{u}^T m'' \hat{v},$$

with $\hat{u}$ and $\hat{v}$ being orthogonal matrices of dimension r and $(r-1)$ respectively, and m'' a diagonal matrix with eigenvalues $(m_1'', \ldots, m_{r-1}'', 0)$. Put

$$\hat{q}^{k\prime} = \hat{v}\hat{e}_k, \quad \hat{p}^{k\prime} = \hat{u}\hat{p}^k. \tag{55}$$

Then α' takes the form

$$\alpha' = \sum_{1}^{r} m_i'' (\hat{p}^{i\prime}, \mathrm{d}\,\hat{q}^{i\prime}), \tag{56}$$

with $\hat{p}_r'$ omitted.

It can be shown as in the previous case that the eigenvalues of the positive matrix $\mathcal{D} = \hat{\mathrm{d}}\,\hat{\mathrm{d}}^T$ satisfy conditions

$$m_i'^{/2} \geq m_i''^{/2} \geq m_{i+1}'^{/2} \tag{57}$$

and now all m_i'' can be chosen positive:

$$m_i' \geq m_i'' \geq m_{i+1}' . \tag{58}$$

Thus after the second step of our descent procedure we obtain the expression

$$\alpha = \sum_{i=1}^{r} m_i' \mathrm{d}\varphi_i' + \sum_{1}^{r-1} m_i'' \mathrm{d}\varphi_i'' + \sum_{1}^{r-1} m_i''(p_i'', \varphi_i''), \tag{59}$$

and now the last term looks exactly as in (34) with the only change $r \to r-1$. We can repeat our procedure once more and so on. Then we get the expression we are looking for:

$$\alpha = \sum_{1}^{r} m_i' \mathrm{d}\varphi_i' + \sum_{1}^{r-1} m_i'' \mathrm{d}\varphi_i'' + \ldots + m_1^{(2r-1)} \mathrm{d}\varphi_1^{(2r-1)} \tag{60}$$

for the case $n = 2r+1$, and

$$\alpha = \sum_{1}^{r} m_i' \mathrm{d}\varphi_i' + \sum_{1}^{r,} m_i'' \mathrm{d}\varphi_i'' + \ldots + m_1^{(2r-2)} \mathrm{d}\varphi_1^{(2r-2)}$$

if $n = 2r$. Here $\varphi_i^{(k)}$ are angle variables $0 \leq \varphi_i^{(k)} < 2\pi$ and the domain of $m_i^{(k)}$ is determined by our recurrent descent procedure

$$m_i^{(k-1)} \geq m_i^{(k)} \geq m_{i+1}^{(k-1)}, \tag{61}$$

but for the last elements, in each line which satisfy

$$m_{i_0}^{(k-1)} \geq m_{i_0}^{(k)} \geq |m_{i_0+1}^{(k-1)}| \qquad \text{if } n-k = 2p+1$$

$$m_{i_0}^{(k-1)} \geq m_{i_0}^{(k)} \geq -m_{i_0}^{(k-1)} \qquad \text{if } n-k = 2p.$$

This is the classical analog of the Gelfand-Zetlin table for the $SO(n)$ algebra.

As in §2, the form (60) defined on the polyhedron (61) times the torus coincides with the Kirillov form on the orbit with some «singular» submanifold deleted.

3. CALCULATION OF THE FUNCTIONAL INTEGRAL

After the introduction of convenient canonical coordinates on the orbit the functional integral for the higher rank can be treated exactly as that in the introduction for the $SO(3)$ case. We introduce the main functional integral in the form

$$G(\{\varphi'\}, \{\varphi''\}) \int_{i,k,t} \prod \mathrm{d}m_i^{(k)} \mathrm{d}\varphi_i^{(k)} \, e^{i\int \left[m_i^{(k)}(t) + \gamma_i^{(k)}\right)\dot{\varphi}_i^{(k)} - h(M(t))\right] \mathrm{d}t} \tag{62}$$

where the hamiltonian $h(M)$ is a function on the orbit and $\gamma_i^{(k)}$ are some constants; we integrate over the phase space trajectories $m_i^{(k)}(t), \varphi_i^{(k)}(t)$ lying within the boundaries (31), (61) for $m(t)$ and $-\infty < \varphi(t) < \infty; 0 < t < T$. The boundary conditions are given by

$$\varphi_i^{(k)}(0) = \varphi_i^{(k)\prime}, \varphi_i^{(k)}(T) = \varphi_i^{(k)\prime\prime}(\mathrm{mod}\ 2\pi). \tag{63}$$

Alternätively, we can fix the boundary conditions completely and average over the φ'' shifted by any integer multiples of 2π.

The condition of single-valuedness of the action leads to intergral orbits which correspond to the cases when the parameters m_i are all integers or half integers. In the former case the action is single-valued for $\gamma_i^{(k)} = 0$. In the latter case we are to take all $\gamma_i^{(k)} = 1/2$.

It is clear that the functional integral (62) can be reduced to the finite-dimensional one if the hamiltonian is any function of $m_i^{(k)} = 0$. In particular, the generators belonging to some Cartan subgroup are specific linear combinations of m-variables. For the $SU(n)$ case the explicit formula looks as follows [11]

$$P^{(k)}(m) = \sum_i m_i^{(k)} = 0$$

The functional integral for fixed boundary condition is reduced to

$$G(\{\varphi'\},\{\varphi''\}) \int \mathrm{d}\,m\, e^{i\sum_{i,k}\left[m_i^{(k)}(\varphi_i^{(k)\prime\prime}-\varphi_i^{(k)\prime})-h(m)T+\gamma_i^{(k)}(\varphi_i^{(k)\prime\prime}-\varphi_i^{(k)\prime})\right]} \tag{64}$$

and the averaging over final φ-variables gives

$$\begin{aligned} G(\{\varphi'\},\{\varphi''\}) &= \sum_{\{n_i^{(k)}\}} G_0(\{\varphi'\},\{\varphi''+2\pi n\}) = \\ &= \int \mathrm{d}\,m\, e^{i\sum_{i,k}\left[m_i^{(k)}+\gamma_i^{(k)})(\varphi_i^{(k)\prime\prime}-\varphi_i^{(k)\prime})-h(m)T\right]}\delta(\alpha,m), \end{aligned} \tag{65}$$

where $\delta(\alpha, m)$ is the periodic δ-function supported at all integer points for $\alpha_i^{(k)} = 0$ and at all half-integer points for $\alpha_i^{(k)} = 1/2$ This δ-function in the integral allows to perform the last integration over m. However, we are to remember the regularization, analogous to that in the introduction (8), namely integration over $m_i^{(k)}$ is to be taken over the domain

$$m_i^{(k+1)} + \epsilon \geq m_i^{(k)} \geq m_i^{(k-1)} - \epsilon \tag{66}$$

with the limit $\epsilon \to 0$.

As a result we shall get an expression of the type

$$G(\{\varphi'\},\{\varphi''\}) = \sum_{m} e^{i\left(\sum\left[m_i^{(k)}+(\varphi_i^{(k)''}-\varphi_i^{(k)'})-h(m)T\right)\right]} \tag{67}$$

where the sum is taken over all quantized Gelfand-Zetlin tables with fixed $m_1, \ldots, m_z$, the latter evidently playing the role of the highest weight of the representation corresponding to the orbit.

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